Questions 1 and 2 can be found in the text of the solution.

3. (25 points) A household is composed of two individuals: m and f. The household derives utility from consumption (c) and from leisure (l). Moreover, since the two members of the household enjoy spending time together, their utility from individual leisure is non-separable:

\[ u(c_t, l_{mt}, l_{ft}) = c_t^{1-\sigma} + (l_{mt}, l_{ft})^{1-\delta} \quad 0<\delta<1 \quad 0<\sigma<1 \]

The household can save or borrow money in an interest-bearing account, receiving interest payments with interest rate r. The two members of household receive different wages for their labor, \( w_{mt} \) and \( w_{ft} \). Since f is a PhD economist and m is lawyer, \( w_{mt} > w_{ft} \) for all t. Both have a total of 365 units of time to allocate between work and leisure in each period, so that the household’s budget constraint is:

\[ a_{t+1} = (1+r)[w_{mt}(365-l_{mt}) + w_{ft}(365-l_{ft}) + a_t - c_t] \]

\[ 0 \leq l_{ft} \leq 365 \quad 0 \leq l_{mt} \leq 365 \]

a) Write the household’s problem as a dynamic programming problem and write the Bellman’s equation. What are the control variables and what are state variables of the problem. Assume that the household makes decisions as a single unit (i.e. there are not separate decisions for f and for m). Assume also that the household lives forever and has a rate of time preference of \( \beta \). Finally, assume that the household knows in advance the time path of future wages for both members of household.

b) Write the first order condition(s) of the problem and the envelope condition(s).

c) Find a decision rule on the relationship between how much time each member of household works, assuming that both work and neither works all the time. Which of the two will work more in this case? Are there conditions under which it is optimal for one member of household to work all the time but for the other to work for less than 365 units of time?

d) Derive an Euler equation showing the time path of consumption (\( c_{t+1} \) as a function of \( c_t \)). What will determine whether consumption is increasing or deceasing over time?

4. (25 points) Consider the following system of differential equations:

\[ x' = y + x(c - x^2 - y^2) \]

\[ y' = -x + y(c - x^2 - y^2) \]
a) Find the steady state of the system. How does it depend on the value of $c$?

b) Linearize the system around the steady state and compute the system’s eigenvalues. How does the stability of the system depend on the value of $c$?
Solutions

Question 3

a) (6 points)

Control : $l_m, l_f, c$ (or $a_{t+1}$)

State: $a_t$

$$V(a_t) = c_t^{1-\sigma} + (l_{mt,l_{ft}})_{1-\delta} + \mu_m l_{mt} + \mu_f l_{ft} + \lambda_m (365 - l_{mt}) + \lambda_f (365 - l_{ft}) + \beta V(a_{t+1})$$

$$= c_t^{1-\sigma} + (l_{mt,l_{ft}})_{1-\delta} + \mu_m l_{mt} + \mu_f l_{ft} + \lambda_m (365 - l_{mt}) + \lambda_f (365 - l_{ft}) + \beta V[(1+r)(w_{mt}(365-l_{mt}) + w_{ft}(365-l_{ft}) + a_t - c_t)]$$

b) (8 points)

FOC

$$(c_t) (1-\sigma) c_t^{-\sigma} = \beta (1 + r)V'(a_{t+1})$$

$$(l_{mt}) (1-\delta) l_{mt}^{1-\delta} + \mu_m - \lambda_m = \beta(1+r)w_{mt} V'(a_{t+1})$$

$$(l_{ft}) (1-\delta) l_{ft}^{1-\delta} + \mu_f - \lambda_f = \beta(1+r)w_{ft} V'(a_{t+1})$$

Envelope:

$$V'(a_t) = \beta(1+r)V'(a_{t+1})$$

c) (6 points)

Dividing the two first order conditions on leisure we get:

$$l_{ft}/l_{mt} = w_{mt} / w_{ft}$$

denoting with $h$ the number of days each member of household works, we have:

$$(365 - h_{ft})/(365 - h_{mt}) = w_{mt} / w_{ft}$$

Since $w_{mt} > w_{ft}$, $h_{mt} > h_{ft}$ for all $t$.

If $m$ worked all the time, but $f$ didn’t, the third FOC would be written as:
0 = \beta w \theta V'(a_{t+1}) + \lambda f - \mu f

Since \ V'(a_{t+1}) > 0 from the first FOC, it must be the case that \mu f > 0, so that both are working all the time.

d) (5 points)

Writing the FOC for one period earlier, we get:

\[ (1 - \sigma) c_{t-1}^{-\sigma} = \beta (1 + r) V'(a_t) \]

Plugging this and the first FOC into the Envelope condition, we get:

\[ c_{t-1}^{-\sigma} / c_t^{-\sigma} = \beta (1 + r) \]

updating this by one period, we obtain the Euler equation:

\[ c_t = [\beta (1 + r)]^{1/\sigma} c_{t+1} \]

So that \( c_t \) will be following an increasing (path) when \( \beta (1 + r) > (<) 1 \)

4. (25 points) Consider the following system of differential equations:

\[ x' = y + x(c - x^2 - y^2) \]
\[ y' = -x + y(c - x^2 - y^2) \]

a) Find the steady state of the system. Does it depend on the value of \( c \)?

(10 points)

Solving for \( x' = 0, y' = 0 \), we obtain:

\[ y = -x(c - x^2 - y^2) \] (1)
\[ x = y(c - x^2 - y^2) \] (2)

Case 1: \( x^2 + y^2 = c \)

\[ y = x = 0 \]

Case 2a: \( x^2 + y^2 \neq c \), \( y \neq 0, x \neq 0 \)

\[ y/x = -x/y \]
Not possible. Thus, either \(x=0\) or \(y=0\)

**Case 2b: \(x^2 + y^2 \neq c, y=0\)**

From equation (2), \(x = 0\)

**Case 2c: \(x^2 + y^2 \neq c, x=0\)**

From equation (1), \(y = 0\)

So that \(x=y=0\) is the steady state regardless of the value of \(c\).

b) (15 points)

\[
\begin{align*}
\frac{\partial x'}{\partial x} |_{ss} &= c - x^2 - y^2 - 2x^2 |_{ss} = c & \frac{\partial x'}{\partial y} |_{ss} &= 1 - 2xy |_{ss} = 1 \\
\frac{\partial y'}{\partial x} |_{ss} &= -1 - 2xy |_{ss} = -1 & \frac{\partial y'}{\partial y} |_{ss} &= c - x^2 - y^2 - 2y^2 |_{ss} = c
\end{align*}
\]

So that our system can be linearized as:

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
c & 1 \\
-1 & c
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Finding the eigenvalues:

\[
\begin{vmatrix}
c - \lambda & 1 \\
-1 & c - \lambda
\end{vmatrix} = 0
\]

\[
\lambda^2 - 2c\lambda + c^2 + 1 = 0
\]

\(
\lambda_{1,2} = c \pm i
\)

Eigenvalues are complex. The system exhibits cycles if \(c=0\). For \(c\neq0\), it exhibits a spiral path: a stable spiral if \(c<0\) and an unstable spiral if \(c>0\). In either case, the spiral is clockwise.