Solutions

1) Consider each of the following, and provide an example (the example may be graphical or algebraic), or cite a theorem or provide a proof as to why no such example can be found.

i) A function that is strictly quasi-concave and strictly quasi-convex, but is neither concave nor convex. (5 points)

Consider a strictly increasing function shaped like an integral sign. Alternatively, consider \( f(x) = x^3 \).

ii) A non-empty, compact set that is open. (5 points)

Impossible. Non-empty rules out the empty set. Compact requires closed and bounded. Bounded rules out any other set without a boundary (ie whose boundary is the empty set, eg the whole real line in \( \mathbb{R}^1 \)). Then, if the set is closed, all points on the boundary of the set are members of the set. But for these points, we cannot draw an open ball without including points not inside the set, and so the set is not open.

iii) Two non-empty convex sets \( C \) and \( D \) such that \( \text{Int}[C] \neq \emptyset \), \( \text{Int}[D] \neq \emptyset \), where \( C \cap B(D) = \emptyset \) (where \( B(D) \) denotes the boundary of \( B \)), but where we cannot find a hyperplane that separates the sets. (5 points)

Suppose \( C \) is entirely contained within the set \( D \). Then this meets the definition (the boundary of \( D \) is only the points on its edges, which do not intersect with \( C \)), but clearly we cannot find a separating hyperplane.

(Note that the boundary of an open set is still all the points around its edge – ie for the set \( A = [1,2) \) the boundary is the set \( B = \{1\}, \{2\} \). So having \( D \) be an open set intersecting with \( C \) does not meet the requirements).

iv) A continuous function \( f: \mathbb{R}^1 \to \mathbb{R}^1 \) that maps any point in some convex, compact set \( X \) into some convex, compact set \( Y \), but does not possess a fixed point. (5 points)

The function here need not map into itself – there is no requirement here that \( X \) and \( Y \) be the same set. So consider the function \( y = x + 0.2 \), over the set \([0,1]\).

v) A continuous, strictly concave function that does not possess a maximum. (5 points)

There is no requirement here for the function to be maximized over a compact set. Consider a strictly concave function that increases forever, such as \( f(x) = \log(x) \).
2) An airline is choosing how many tickets to sell on a particular airline route. The airline faces demand \( P(q) = A - bq \), where \( A \) and \( b \) are strictly positive. The airline uses a plane with a maximum capacity of 80 passengers, and cannot sell negative tickets. The cost for a flight is \( c \) per person, with a sunk cost \( F \) for running the flight (ie the cost must be paid even if \( q = 0 \)).

a) Characterize the airline’s problem as a constrained optimization problem. (3 points)

\[
\text{Max}_q q[A - bq - c] - F \quad \text{s.t. } q \geq 0, \ 80 - q \geq 0
\]

b) Write down the Lagrangean and the first order condition(s) and complementary slackness condition(s). (6 points)

\[
L: q[A - bq - c] - F + \lambda(80 - q)
\]

Focs:
\[
q: A - 2bq - c - \lambda \leq 0 \\
\lambda: 80 - q \geq 0
\]

CS:
\[
q: (A - 2bq - c - \lambda)q = 0 \\
\lambda: (80 - q)\lambda = 0
\]

Note that \( A \) and \( b \) are parameters, not choice variables; do not try and include multipliers/take FOCs/CS conditions for them.

c) Under what condition(s) will the airline sell all of its seats, and what price will it charge when it does so and what profit will it make? Under what condition(s) will the airline sell none of its seats, what price will it charge when it does so, and what profit will it make? Under what condition(s) will the airline choose some intermediate quantity of seats, what price will it charge if it does so, and what profit will it make? (12 points)

Case 1: Right boundary, \( q = 80 \).
From first CS condition, \( \lambda = A - 160b - c \)
And we know \( \lambda \geq 0 \) from how we set the problem up. So we must have \( A \geq 160b + c \).
Price = \( A - 80b \)
Profit = \( 80A - 6400b - 80c - F \)

Case 2: Left boundary, \( q = 0 \)
From the second CS, we have \( \lambda = 0 \).
From the first FOC, we must have \( A \leq c \).
Price = \( A \)
Profit = \( - F \)

Case 3: Internal solution, \( 0 < q < 80 \)
From the second CS, we have that \( \lambda = 0 \).
From the first CS, we have that \( A - 2bq - c = 0 \), and so \( q = (A - c)/2b \)
Price = \( A - (A - c)/2 = (A + c)/2 \)
Profit = q(p - c) - F
= \left( \frac{A - c}{2b} \left( \frac{A + c}{2} - c \right) \right) - F
= \left( \frac{A - c}{2b} \left( A + c - 2c \right) \right) - F
= \frac{(A - c)(A - c)}{4b} - F
= \frac{(A - c)^2}{4b} - F

For this to be optimal, we need \( A \geq c \), and \( A \leq 160b + c \)

\( d \) Now suppose that the fixed cost is no longer sunk. That is, if the airline chooses \( q = 0 \), it does not have to run the flight, and does not incur costs \( F \). How does this change the answer above?

The only difference is that now \( q = 0 \) is more attractive than it otherwise was.

\{ This changes the conditions we need to be in each case, so it is not correct to say that the FOCs are the same (nothing changes at the margin) and so the solution is unchanged - that only applies for continuous functions, and now our profit function is discontinuous at \( q = 0 \). \\
Eg: for some ranges of \( A, b, c \), previously we would have chosen \( q > 0 \) but now we choose \( q = 0 \) instead. \}

Specifically:
If we were in the \( A \geq c \) case already, then clearly \( q = 0 \) remains optimal.
If we were in the internal solution where \( c \leq A \leq 160b + c \), then we should now switch to \( q = 0 \) only if \( (A - c)^2 / 4b < F \), and this would increase our profit to 0.
If we were on the right constraint where \( q = 80 \), we should switch to \( q = 0 \) only if \( 80A - 6400b - 80c < F \).

Question 3

a) (6 points)

Control: \( l_m, l_f, c \) (or \( a_{t+1} \))
State: \( a_t \)

\[ V(a_t) = c_t^{1-\sigma} + (l_{mt}l_{ft})^{1-\delta} + \mu_m l_{mt} + \mu_f l_{ft} + \lambda_m (365 - l_{mt}) + \lambda_f (365 - l_{ft}) + \beta V(a_{t+1}) \]
\[ c_t^{1-\sigma} + (l_{m,t}l_{f,t})^{1-\delta} + + \mu_ml_{mt} + \mu_fl_{ft} + \lambda_m(365 - l_{mt}) + \lambda_f(365 - l_{ft}) + \beta V[(1+r)(w_{mt}(365-l_{mt}) + w_{ft}(365-l_{ft}) + a_t - c_t)] \]

b) (8 points)

FOC

\[ (c_t) (1- \sigma)c_t^{-\sigma} = \beta (1 + r)V'(a_{t+1}) \]

\[ (l_{m,t}) (1- \delta)l_{mt}^{-\delta} l_{ft}^{1-\delta} + \mu_m - \lambda_m = \beta(1+r)w_{mt}V'(a_{t+1}) \]

\[ (l_{f,t}) (1- \delta)l_{ft}^{-\delta} l_{mt}^{1-\delta} + \mu_f - \lambda_f = \beta(1+r)w_{ft}V'(a_{t+1}) \]

Envelope:

\[ V'(a_t) = \beta(1+r)V'(a_{t+1}) \]

c) (6 points)

Dividing the two first order conditions on leisure we get:

\[ \frac{l_{ft}}{l_{mt}} = \frac{w_{mt}}{w_{ft}} \]

denoting with \( h \) the number of days each member of household works, we have:

\[ \frac{(365 - h_{ft})}{(365 - h_{mt})} = \frac{w_{mt}}{w_{ft}} \]

Since \( w_{mt} > w_{ft} \), \( h_{mt} > h_{ft} \) for all \( t \).

If \( m \) worked all the time, but \( f \) didn’t, the third FOC would be written as:

\[ 0 = \beta w_{ft}V'(a_{t+1}) + \lambda_f - \mu_f \]

Since \( V'(a_{t+1}) > 0 \) from the first FOC, it must be the case that \( \mu_f > 0 \), so that both are working all the time.

d) (5 points)

Writing the FOC for one period earlier, we get:

\[ (1- \sigma)c_{t-1}^{-\sigma} = \beta (1 + r)V'(a_t) \]

Plugging this and the first FOC into the Envelope condition, we get:
\[ c_{t-1}^{-\sigma} / c_t^{-\sigma} = \beta (1 + r) \]

updating this by one period, we obtain the Euler equation:

\[ c_t = \beta(1 + r)]^{1/\sigma} c_{t+1} \]

So that \( c_t \) will be following an increasing (path) when \( \beta(1 + r) > (<) 1 \)

4. (25 points) Consider the following system of differential equations:

\[ x' = y + x(c - x^2 - y^2) \]
\[ y' = -x + y(c - x^2 - y^2) \]

a) Find the steady state of the system. Does it depend on the value of c?

(10 points)
Solving for \( x' = 0, y' = 0 \), we obtain:

\[ y = - x(c - x^2 - y^2) \] (1)
\[ x = y(c - x^2 - y^2) \] (2)

Case 1: \( x^2 + y^2 = c \)
\( y = x = 0 \)

Case 2a: \( x^2 + y^2 \neq c, y \neq 0, x \neq 0 \)
\( y/x = -x/y \)
Not possible. Thus, either \( x=0 \) or \( y=0 \)

Case 2b: \( x^2 + y^2 \neq c, y=0 \)
From equation (2), \( x = 0 \)

Case 2c: \( x^2 + y^2 \neq c, x=0 \)
From equation (1), \( y = 0 \)
So that \( x=y=0 \) is the steady state regardless of the value of c.

b) (15 points)
\[
\frac{\partial x'}{\partial x} |_{ss} = c - x^2 - y^2 - 2x^2 |_{ss} = c \quad \frac{\partial x'}{\partial y} |_{ss} = 1 - 2xy |_{ss} = 1 \\
\frac{\partial y'}{\partial x} |_{ss} = -1 - 2xy |_{ss} = -1 \quad \frac{\partial y'}{\partial y} |_{ss} = c - x^2 - y^2 - 2y^2 |_{ss} = c
\]

So that our system can be linearized as:

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
c & 1 \\ -1 & c
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Finding the eigenvalues:

\[
\begin{vmatrix}
c - \lambda & 1 \\ -1 & c - \lambda
\end{vmatrix} = 0
\]

\[\lambda^2 - 2c\lambda + c^2 + 1 = 0\]

\[\lambda_{1,2} = c \pm i\]

Eigenvalues are complex. The system exhibits cycles if \(c=0\). For \(c\neq 0\), it exhibits a spiral path: a stable spiral if \(c<0\) and an unstable spiral if \(c>0\). In either case, the spiral is clockwise.