1. (10 points) A consumer lives forever in discrete time. They choose consumption $c_t$, assets $a_{t+1}$, and labor supply $l_t$. The interest rate is constant, but wages may fluctuate deterministically over time. The consumer’s Lagrangian is

$$
\max_{a_t, c_t, l_t} \sum_{t=0}^{\infty} \beta^t \left[ U(c_t, l_t) + \lambda_t \left( R \left( a_t + w_t l_t - c_t \right) - a_{t+1} \right) \right] \tag{1}
$$

where $w_t$ is the wage and $R$ is the gross interest rate.

(a) Write down the Bellman equation for this problem. For this part only, you may assume wages are constant over time. What are the state and control variables?

**ANSWER:**

$$
V(a_t) = \max_{c_t, l_t} U(c_t, l_t) + \beta V \left( R \left( a_t + w_t l_t - c_t \right) \right) \tag{2}
$$

The endogenous state variable is assets $a_t$. $c_t$ and $l_t$ are control variables. If wages are not constant then calendar time $t$ must be included as a state variable.

Using any method you like, solve for the following. Eliminate any Lagrange multipliers from the final expressions:

(b) An intertemporal Euler equation relating $U_c(c_t, l_t)$ to $U_c(c_{t+1}, l_{t+1})$.

**ANSWER:** The FOC for the Lagrangian are

$$
U_c(c_t, l_t) = R\lambda_t \tag{3}
$$

$$
U_l(c_t, l_t) = -R\lambda_t w_t \tag{4}
$$

$$
\lambda_t = \beta R\lambda_{t+1} \tag{5}
$$

Using the the consumption and asset FOCs, I have

$$
U_c(c_t, l_t) = \beta RU_c(c_{t+1}, l_{t+1}). \tag{6}
$$

(c) An intratemporal condition relating $U_c(c_t, l_t)$ to $U_l(c_t, l_t)$. 


(d) An intertemporal Euler equation relating $U_l(c_t, l_t)$ to $U_l(c_{t+1}, l_{t+1})$.

**Answer:** Using the consumption and labor FOCs I have

\[
\frac{U_l(c_t, l_t)}{U_c(c_t, l_t)} = -w_t \tag{7}
\]

For the rest of the question assume that $\beta R = 1$, and that the utility function has the form

\[
U(c, l) = f(c) - l^\alpha, \tag{10}
\]

where $f' > 0$, $f'' < 0$ and $\alpha > 1$.

(e) Using your answer in part (d), show how $l_{t+1}$ relates to $l_t$ as a function of $w_t$ and $w_{t+1}$.

**Answer:** Plugging in the functional form and $\beta R = 1$, I have

\[
\alpha l_t^{\alpha - 1} = \alpha \beta R \frac{w_t}{w_{t+1}} l_{t+1}^{\alpha - 1} \tag{11}
\]

\[
l_t^{\alpha - 1} = \frac{w_t}{w_{t+1}} l_{t+1}^{\alpha - 1} \tag{12}
\]

\[
l_t \left( \frac{w_{t+1}}{w_t} \right)^{\frac{1}{\alpha - 1}} = l_{t+1} \tag{13}
\]

(f) Given the assumptions above, what do we know about the time path of consumption $c_t$? Will $c_t$ depend directly on the current wage $w_t$?

**Answer:** Under the assumptions given, the consumption Euler equation is

\[
U_c(c_t, l_t) = \beta R U_c(c_{t+1}, l_{t+1}) \tag{14}
\]

\[
f'(c_t) = f'(c_{t+1}) \tag{15}
\]

Because $f'$ is one-to-one, the Euler equation implies $c_t = c_{t+1}$ for all $t$, so the consumption path is flat.

(g) Now assume the utility function is $U(c, l) = c_t^\gamma (H-l)^{1-\gamma}$, where $\gamma \in (0, 1)$ and $H$ is the total time endowment. Do your conclusions in part (f) change? Why or why not? Remember, the wage may vary over time.

**Answer:** The new utility function is not additively separable, so variation in $l_t$ changes the marginal utility of consumption. This means that wage fluctuations will cause variation in the consumption path. Formally,

\[
U_c = \gamma c_t^{\gamma - 1}(H - l_t)^{1-\gamma} \tag{16}
\]
So the consumption Euler equation becomes

$$\gamma c_t^{\gamma-1} (H - l_t)^{1-\gamma} = \gamma c_{t+1}^{\gamma-1} (H - l_{t+1})^{1-\gamma}$$  \hspace{1cm} (17)$$

$$\left( \frac{c_t}{c_{t+1}} \right)^{\gamma-1} = \left( \frac{H - l_{t+1}}{H - l_t} \right)^{1-\gamma}$$  \hspace{1cm} (18)$$

We can see that whenever $l_t$ varies over time, $c_t$ must vary as well. So the consumption time path is no longer flat: (temporary) wage fluctuations cause consumption to change when utility is non-separable.

2. Consider the first order, scalar difference equation

$$x_{t+1} = x_t^3$$  \hspace{1cm} (19)$$

(a) Plot $x_{t+1}$ as a function of $x_t$.

**ANSWER:** It is just the function $y = x^3$

(b) What are the steady states of this process?

**ANSWER:** The steady states satisfy

$$x = x^3$$  \hspace{1cm} (20)$$

Evidently, the steady states are $\{-1, 0, 1\}$

(c) Which steady states are (asymptotically) stable?

**ANSWER:** Only $x_t = 0$ is stable. There is no open neighborhood of $x_t = 1$ where the process converges to 1, likewise for $x_t = -1$.

3. Consider the scalar difference equation

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2}$$  \hspace{1cm} (21)$$

(a) Depending on the values $\rho_1, \rho_2$, what are the steady states of this process?

**ANSWER:** The steady states satisfy

$$x = \rho_1 x + \rho_2 x$$  \hspace{1cm} (22)$$

$$x - \rho_1 x - \rho_2 x = 0$$  \hspace{1cm} (23)$$

so whenever $1 \neq \rho_1 + \rho_2$, the steady state is

$$x = \frac{1}{1 - \rho_1 - \rho_2}$$  \hspace{1cm} (24)$$

When $1 = \rho_1 + \rho_2$, I have $\rho_1 x + \rho_2 x = x$, so the steady state equation becomes

$$x = x.$$

\hspace{1cm} (25)
Thus, every point is a steady state in this case.

(b) Write the process as a first-order vector difference equation.

**ANSWER:** Let

\[ x_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \]  

(26)

And

\[ A = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \]  

(27)

Then

\[ x_t = Ax_{t-1} \]  

(28)

describes the original system.