1. MIU

Consider the household saving problem with two assets: assets \(a\) that earn a real return \(r\), and money \(m\). Although money does not earn a return, it is useful in transactions. We express this by inserting money into the utility function (this is called money-in-utility model, of MIU). The dynamic problem becomes

\[
\max_{\{m,c\} : a_0 = 0} \sum_{t=0}^{\infty} \beta^t [u(c_t) + \phi(m_t)]
\]

subject to the budget constraint

\[
a_{t+1} = (1 + r)(a_t + y_t + m_t - c_t - m_{t+1})
\]

Suppose that \(u(c)\) and \(\phi(m)\) are both increasing, concave, and satisfy the Inada conditions (i.e. \(u_c, \phi_m \to \infty\) as \(c, m \to 0\), \(u_c, \phi_m \to 0\) as \(c, m \to \infty\)). (note: in the questions below, “interpret” means provide economic intuition for the result).

1. What are the state variables? What are the control variables?

   (a) The state variables are \(a_t\) and \(m_t\). The control variables are \(c_t\), and also \(a_{t+1}\) and \(m_{t+1}\).

2. Formulate this problem as a dynamic programming problem and write the Bellman equation.

   (a) The problem is

   \[
   V(m, a) = \max_{c, m'} \{u(c) + \phi(m) + \beta V(m', a')\}
   \]

   where \(a' = (1 + r)(a + y + m - c - m')\). (i.e. we substitute the transition equation into \(V\) for \(a'\)).

3. Compute the first-order condition(s) of the problem. Compute the envelope condition(s) of the problem.

   (a) FOC’s w.r.t. \(c\) and \(m'\) are

   \[
   u'(c) = \beta (1 + r)V_{a'} \\
   V_{m'} = (1 + r)V_{a'}
   \]

   (b) Envelope conditions are

   \[
   V_a = \beta (1 + r)V_{a'} \\
   V_m = \phi'(m) + \beta (1 + r)V_{a'}
   \]

4. Combine the first-order and envelope conditions to derive the standard consumption euler equation. Interpret.

   (a) Substituting the \(c\) FOC into the \(a\) EC yields \(V_a = u'(c)\). Subbing this into the \(a\) EC yields the consumption euler equation

   \[
   u'(c) = \beta (1 + r)u'(c')
   \]

   (b) This says that the marginal utility of consumption today must equal the interest rate times the MU tomorrow. Intuitively, at the solution the households must be indifferent between consuming the marginal unit today and saving it, earning return \(1 + r\), and consuming it tomorrow.

5. Derive an expression relating \(c\) and \(m\). Interpret.
(a) Subbing the c FOC into the m EC yields $V_m = \phi'(m) + u'(c)$. Iterating this and subbing into the $m'$ FOC yields $\phi'(m') + u'(c') = (1 + r) V_{u'}$. Decrementing and replacing $V_u = u'(c)$ yields

$$\phi'(m) = ru'(c)$$

(b) This reflects the tradeoff between holding money or storing that wealth as assets that earn interest. A unit of money yields marginal return $\phi'(m)$, and then can be used to purchase one unit of goods as well. However, that money could alternatively have been invested, in which case it would have yielded a return of $1 + r$, allowing the purchase of an additional $r$ units of goods today. Thus the price of money in terms of goods is the interest rate $r$

6. What happens to $m$ as $r \to 0$? Why can we never have $r \leq 0$? Interpret. (note: this property is called the zero lower bound).

(a) As $r \to 0$, we have $\phi'(m) \to 0$, and from the Inada conditions $m \to \infty$.

(b) Intuitively, $r$ is the cost of money, and as money becomes infinitely cheap, its consumption will approach infinity.

(c) Observe that if $r = 0$, money and assets yield the same return. As long as money has positive marginal benefit $\phi'(m) > 0$, households will strictly prefer money to assets. Therefore at $r = 0$ households will borrow an infinite quantity of assets in order to purchase an infinite quantity of money.

2. Phillips Curve

Consider the following differential equation that describes how unemployment $u(t)$ and expected inflation $\pi^e(t)$ evolve over time for a sequence of actual inflation rates $\pi(t)$

$$u(t) = \rho (\pi - u(t)) + \kappa (\pi^e(t) - \pi(t))$$

$$\pi^e(t) = \gamma (\pi(t) - \pi^e(t))$$

The initial expected inflation and unemployment satisfy $\pi^e(0) = \pi^e_0$ and $u(0) = u_0$.

1. Compute an expression for $\pi^e(t)$ as a function of $\pi^e_0$ and past realizations of $\pi(t)$. Interpret.

(a) We have $\pi^e(t) + \gamma \pi^e(t) = \gamma \pi(t)$. Multiplying both sides by integrating factor $e^\gamma t$, we obtain $e^\gamma t \pi^e(t) + \gamma e^\gamma t \pi^e(t) = e^\gamma t \gamma \pi(t)$, which by the product rule is $(e^\gamma t \pi^e(t))^1 = e^\gamma t \gamma \pi(t)$. Integrating both sides $\int_0^t e^\gamma s \gamma \pi(s) ds$, which we can write as

$$\pi^e(t) = e^{-\gamma t} \left( \pi^e_0 + \gamma \int_0^t e^\gamma s \pi(s) ds \right)$$

(b) Expected inflation is a weighted average of past inflation, with greater weight placed on the recent past. In the limit as $t \to \infty$, the effect of the initial inflation expectations goes to zero.

2. Suppose that the Fed chooses to stabilize inflation $\pi(t)$ at some fixed rate $\overline{\pi}$. Compute expressions for $\pi^e(t)$. Interpret.

(a) Letting $\pi (s) = \overline{\pi}$ in our expression above, we obtain $\pi^e(t) = e^{-\gamma t} \left( \pi^e_0 + \gamma \int_0^t e^\gamma s \pi(s) ds \right)$. Further simplification yields

$$\pi^e(t) = \overline{\pi} + e^{-\gamma t} (\pi^e_0 - \overline{\pi})$$

(b) When $t = 0$, this expression is $\pi^e(t) = \pi^e_0$. As $t \to \infty$, $\pi^e \to \overline{\pi}$. Expected inflation converges smoothly to actual inflation.

3. Compute an expression for $u(t)$. Interpret.
(a) Subbing the expression for $\pi'(t)$ into the differential equation for $u$, we obtain

$$u'(t) + \rho u(t) = \rho \pi + \kappa e^{-\gamma t} (\pi_0^e - \pi)$$

Multiplying both sides by the integrating factor $e^{\rho t}$, we obtain $(e^{\rho t}u(t))' = e^{\rho t} \left[ \rho \pi + \kappa e^{-\gamma t} (\pi_0^e - \pi) \right]$. Integrating both sides $\int_0^t$, we obtain

$$\left( e^{\rho s} u(s) \right)'_0 = \int_0^t e^{\rho s} \left[ \rho \pi + \kappa e^{-\gamma s} (\pi_0^e - \pi) \right] ds$$

Further simplification yields

$$u(t) = e^{-\rho t} u_0 + \pi \left( 1 - e^{-\rho t} \right) + \kappa \left( \pi_0^e - \pi \right) \left( \frac{1}{\rho - \gamma} \right) (e^{-\gamma t} - e^{-\rho t})$$

(b) The initial term $e^{-\rho t}u_0 + \pi (1 - e^{-\rho t})$ gives smooth convergence of unemployment from its initial quantity $u_0$ to its long-run level $\pi$. The second term $\kappa \left( \pi_0^e - \pi \right) \left( \frac{1}{\rho - \gamma} \right) (e^{-\gamma t} - e^{-\rho t})$ expresses the fact that this rate of convergence will be slowed by adjustment of inflation expectations. The term $\left( \frac{1}{\rho - \gamma} \right) (e^{-\gamma t} - e^{-\rho t}) > 0$, so convergence will be faster if inflation expectations are initially above the Fed’s inflation rate, and will be slower if inflation expectations are initially below the Fed’s inflation rate.