1. Consider the arbitrage condition for the stock market

\[ \frac{d_{t+1} + p_{t+1} - p_t}{p_t} = r \]

that is, the total rate of return on a stock from \( t \) to \( t + 1 \) (which is dividend yield \( \frac{d_{t+1}}{p_t} \) plus capital gains \( \frac{p_{t+1} - p_t}{p_t} \)) is equal to the required return \( r \). Assume that \( r > 0 \) and that \( 0 < d_t \leq M \) for all \( t \).

(a) Show that the forward solution

\[ p_t^F = \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \]

is a solution to the arbitrage condition. (This solution is sometimes called the fundamental solution.)

**Answer:** Plugging in \( p_t^F = \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \), we obtain

\[ \frac{d_{t+1} + \sum_{i=1}^{\infty} \frac{d_{t+i+1}}{(1 + r)^i} - \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i}}{\sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i}} = r \]

Letting \( \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} = (1 + r) \sum_{i=2}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \), we obtain

\[ \frac{d_{t+1} + \sum_{i=2}^{\infty} \left( (1 + r) \frac{d_{t+i}}{(1 + r)^i} - \frac{d_{t+i+1}}{(1 + r)^i} \right) - \frac{d_{t+i}}{(1 + r)^i}}{\sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i}} = r \]

Multiplying by \( \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \) and simplifying, we obtain

\[ d_{t+1} + \sum_{i=2}^{\infty} \left( \frac{r}{(1 + r)^i} \right) d_{t+i} - \frac{d_{t+1}}{1 + r} = r \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \]

\[ \left( \frac{r}{1 + r} \right) d_{t+1} + \sum_{i=2}^{\infty} \left[ \frac{r}{(1 + r)^i} \right] d_{t+i} = r \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \]

\[ r \sum_{i=1}^{\infty} \left[ \frac{d_{t+i}}{(1 + r)^i} \right] = r \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} \]

(b) Show that

\[ p_t = c(1 + r)^t + p_t^F \]

with \( c \geq 0 \) is also a solution. (The component \( c(1 + r)^t \) is sometimes called the bubble term.)

**Answer:** Substituting this term into the expression

\[ d_{t+1} + p_{t+1} - p_t = r p_t \]

we obtain

\[ d_{t+1} + [c(1 + r)^{t+1} + p_t^F] - [c(1 + r)^t + p_t] = r [c(1 + r)^t + p_t^F] \]
Since \( p_t^F \) is a solution, we have

\[
d_{t+1}^t + p_{t+1}^F - p_t^F = rp_t^F
\]

Canceling this terms, we obtain

\[
c(1 + r)^{t+1} - c(1 + r)^t = rc(1 + r)^t
\]

\[
c(1 + r)(1 + r)^t - c(1 + r)^t = rc(1 + r)^t
\]

\[
(1 + r) - 1 = r
\]

which is true.

(c) Show that the transversality condition \( \lim_{t \to \infty} (1 + r)^{-t} p_t = 0 \) holds if and only if \( c = 0 \).

Answer: Consider

\[
\lim_{t \to \infty} (1 + r)^{-t} p_t = \lim_{t \to \infty} (1 + r)^{-t} \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1 + r)^i} + c(1 + r)^t
\]

\[
= c + \lim_{t \to \infty} (1 + r)^{-t} \sum_{i=1}^{\infty} M \left( \frac{1}{1 + r} \right)^i
\]

\[
\leq c + \lim_{t \to \infty} (1 + r)^{-t} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i \lim_{t \to \infty} (1 + r)^{-t}
\]

\[
= c + \frac{M}{r} \lim_{t \to \infty} (1 + r)^{-t}
\]

\[
= c
\]

2. Reconsider the Solow-Swan growth model that we have discussed in the lectures, and relax two of the assumption we made, namely that the capital stock does not depreciate, and that there is no technical progress.

To allow for depreciation, specify the law of motion for the capital stock as

\[
K^1(t) = sY(t) - \delta K(t)
\]

for \( s, \delta \in (0, 1) \). Let population \( (L(t)) \) grow at the proportional rate \( n \). To allow for technical progress, suppose that the production function is given by

\[
Y(t) = F[K(t), A(t)L(t)]
\]

\[
= [K(t)]^\alpha [e^{\mu t}L(t)]^{1-\alpha}
\]

for \( \mu > 0, \alpha \in (0, 1) \).

(a) First derive the proportional growth rates for \( A, L, K \).

Answer: We have

\[
\frac{A'(t)}{A(t)} = \frac{\mu e^{\mu t}}{e^{nt}} = \mu
\]

\[
\frac{L'(t)}{L(t)} = \frac{n e^{nt}}{e^{nt}} = n
\]

Finally, we have \( K'(t) = sY(t) - \delta K(t) \). Using \( sY(t) = s[K(t)]^\alpha [A(t)L(t)]^{1-\alpha} \), we obtain

\[
\frac{K'(t)}{K(t)} = s\left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha} - \delta
\]

\[
= s\hat{k}(t)^{\alpha-1} - \delta
\]
Derive the implied law of motion for the effective capital-labor ratio, \( \ddot{k}(t) = \frac{K'(t)}{A(t)L(t)} \).

**Answer:** Note that \( \mu \) is the growth rate of the economy. Let \( \ddot{k}(t) = \frac{K'(t)}{A(t)L(t)} \). Then using the quotient rule we have

\[
\ddot{k}'(t) = \frac{K'(t)A(t)L(t) - [A'(t)L(t) + A(t)L'(t)]K(t)}{[A(t)L(t)]^2}
\]

Rearranging, this is

\[
\ddot{k}'(t) = \frac{K'(t)}{K(t)} \left( \frac{K(t)}{A(t)L(t)} \right) - \frac{A'(t)}{A(t)} \left( \frac{K(t)}{A(t)L(t)} \right) - \frac{L'(t)}{L(t)} \left( \frac{K(t)}{A(t)L(t)} \right)
\]

which we can write as

\[
\ddot{k}'(t) = \frac{K'(t)}{K(t)} \left( \frac{K(t)}{A(t)L(t)} \right) - \frac{A'(t)}{A(t)} \left( \frac{K(t)}{A(t)L(t)} \right) - \frac{L'(t)}{L(t)} \left( \frac{K(t)}{A(t)L(t)} \right)
\]

Which is a standard expression saying that percentage growth of \( \ddot{k} \) is the sum of percentage growth in its multiplicative components. Substituting into the expression above, we have

\[
\frac{\ddot{k}'(t)}{k(t)} = \frac{K'(t)}{K(t)} - \frac{A'(t)}{A(t)} - \frac{L'(t)}{L(t)}
\]

Which is the steady state. We can draw a phase diagram with \( \ddot{k} \) on the x-axis, and the curve \( s\ddot{k}\alpha \) and the line \((n + \delta + \mu)\ddot{k}\). Since \( s\ddot{k}\alpha \) satisfies the Inada conditions, there is guaranteed to be a nontrivial intersection of these lines. Moreover, since the curve \( s\ddot{k}\alpha \) lies above the line when \( k < k^{ss} \), and then cuts it from above and lies below the line for \( k > k^{ss} \), we have a convergent steady state.

(c) Compute the (nontrivial) steady state for the effective capital-labor ratio and analyze its stability properties.

**Answer:** The steady state defined implicitly as \( k^{ss} \) that satisfy \( \ddot{k}'(t) = 0 \). Plugging into the differential equation above, we obtain

\[
s [k^{ss}]^{\alpha - 1} = -\delta - \mu - n
\]

\[
\ddot{k}'(t) = s [\ddot{k}(t)]^\alpha - (\delta + \mu + n) \ddot{k}(t)
\]

We can express as

\[
k^{ss} = \left( \frac{s}{\delta + \mu + n} \right)^{\frac{1}{\alpha}}
\]

Which is the steady state. We can draw a phase diagram with \( \ddot{k} \) on the x-axis, and the curve \( s\ddot{k}\alpha \) and the line \((n + \delta + \mu)\ddot{k}\). Since \( s\ddot{k}\alpha \) satisfies the Inada conditions, there is guaranteed to be a nontrivial intersection of these lines. Moreover, since the curve \( s\ddot{k}\alpha \) lies above the line when \( k < k^{ss} \), and then cuts it from above and lies below the line for \( k > k^{ss} \), we have a convergent steady state.