1. Consider the problem of maximizing utility over two goods, $x_1$ and $x_2$, where utility is given by the function

$$U(x_1, x_2) = k x_1^\alpha x_2^\beta,$$

where $\alpha, \beta \in (0, 1)$.

(a) [3 points] Under what conditions is $U(\cdot)$ homogenous of degree one in $(x_1, x_2)$?

The consumer has exogenous income given by $y$, and prices for $x_1$ and $x_2$ are $p_1$ and $p_2$ respectively. There are no other goods in this economy, nor are there opportunities for saving.

Recall that any increasing transformation of the objective function will yield the same optimal choices of $x_1$ and $x_2$. Take the increasing transformation $\ln U(x_1, x_2) - \ln k$ before attempting parts (b)-(f).

(b) [5 points] Write down the constrained optimization problem of maximizing the consumer’s utility subject to a budget constraint as carefully as possible.

(c) [7 points] Prove that there exists a unique solution to the problem that you wrote down in part (b) without first solving the problem.

(d) [10 points] Carefully write down the Lagrangean that corresponds to this problem. Write down the first-order necessary conditions and complementary slackness conditions that result from this consumer’s problem. Provide an argument for which conditions will bind and which will be slack.

(e) [5 points] Write the demand functions $x_1(p, y)$ and $x_2(p, y)$ that solve the constrained utility maximization problem and prove that the demand functions are homogeneous of degree zero.

(f) [5 points] Write the optimal (transformed) utility of the consumer as a function of prices and income using the transformed utility function and your solution to part (e). Take the third- and fourth-order leading principal minors of the bordered Hessian of the transformed utility function. What can you say about the quasiconcavity/quasiconvexity of this function? (In other words, is the function quasiconcave, quasiconvex, possibly quasiconcave, possibly quasiconvex, none of the above, some combination of the above?)
2. [10 points] Prove the following proposition using the envelope theorem:

**Proposition 1 (Roy’s Identity).** Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is continuous and strictly quasiconcave. Let $v(p,y)$ satisfy:

$$v(p,y) = \max_{x \in \mathbb{R}^n} u(x)$$

subject to $p \cdot x \leq y$

$$x_i \geq 0, \forall i = 1, 2, \ldots, n,$$

where $p \in \mathbb{R}^n$ is a vector of non-negative prices. Furthermore assume that for some point $(\bar{p}, \bar{y}) \in \mathbb{R}^{n+1}$ where $\bar{p}_i > 0 \ \forall i = 1, 2, \ldots, n$, $\bar{y} > 0$, $v(\bar{p}, \bar{y})$ is continuously differentiable.

Then for all $i = 1, 2, \ldots, n$,

$$x_i(\bar{p}, \bar{y}) = -\frac{\partial v(\bar{p}, \bar{y})/\partial p_i}{\partial v(\bar{p}, \bar{y})/\partial y}.$$

3. Parts (a)-(d) are unrelated

(a) [5 points] Prove that every finite set is compact. (Note: This set is not necessarily a subset of $\mathbb{R}^n$.)

(b) [5 points] An point $x$ is said an *accumulation point* of the set $S$ if every open ball around $x$ contains elements of $S$ other than $x$. Note that $x$ does not need to be an element of $S$. Prove that if $S$ is closed, any accumulation point of $S$ lies within $S$.

(c) [5 points] Is the function $2 \ln x + 3 \ln y$ a homothetic function? Why or why not?

(d) [5 points] Prove that any intersection of convex sets is convex.

4. Consider the problem where consumers live for $T$ periods and maximize their utility as a function of consumption $c_t$ and healthcare consumption $h_t$. Consumers receive an initial endowment of wealth $a_{-1}$ in period $t = -1$, and can invest in an one-period asset $a_t$ in each subsequent period that yields return $(1 + r)$. The price of consumption is set equal to one, while healthcare consumption costs $p_t$ per unit in each period.

This problem can be written as the following dynamic optimization problem:

$$\max_{\{c_t, h_t\}_{t=0}^T} \sum_{t=0}^{T} \frac{1}{\alpha_t} (h_t^{\alpha_t} + c_t^{\alpha_t})$$

subject to $a_t = (1 + r)a_{t-1} - c_t - p_t h_t$,

$a_T = 0, a_{-1}$ given,

$c_t \geq 0, h_t \geq 0$, 

where $p_t \geq 0$. 

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where $\alpha \in (-\infty, 1)$ and $p_t$ is assumed to be positive. Furthermore, assume that $r > 0$ and $(1 + r)\beta = 1$.

(a) [5 points] Specify state and control variables and write down the Bellman equation.

(b) [7 points] Write down the first-order conditions and envelope conditions for the Bellman equation that you wrote down in part (a).

(c) [7 points] Derive the Euler equation relating $c_t$ to $c_{t+1}$, showing that $c_t$ is some constant $\bar{c}$ over time. Additionally, derive the Euler equation relating $h_t$ to $c_t$.

(d) [10 points] Use your solution to rewrite the household’s budget constraint as a first-order non-autonomous difference equation in $a$. Use the boundary conditions to solve for $\bar{c}$ as a function of model parameters and the sequence of prices. (Note: there are other ways to solve for $\bar{c}$ as a function of model parameters and prices. You must solve the difference equation to receive full credit.)

(e) [6 points] How would a change in price in period $t$ impact the consumption decision $\bar{c}$ in period zero? How does this relationship depend on $\alpha$?