1. Linear Systems

In this section we will study systems of linear differential equations. Without loss of generality, we will restrict our study to first-order systems:

\[
\begin{align*}
\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} &= \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}
\end{align*}
\]

1.1. Uncoupled Systems (The Matrix \( A \) is Diagonal). Suppose that \( A \) is a diagonal matrix:

\[
A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}
\]

Then, we can solve each of the \( n \) equations separately. For instance, the first equation is:

\[
x_1(t) = a_{11}x_1(t) + b_1
\]

and as we have seen, its solution is:

\[
x_1(t) = ce^{a_{11}t} - \frac{b_1}{a_{11}}
\]

and if we have an initial condition:

\[
x_1(0) = x_{10}
\]

then solution becomes:

\[
x_1(t) = \left( x_{10} + \frac{b_1}{a_{11}} \right) e^{a_{11}t} - \frac{b_1}{a_{11}}
\]

Therefore, if we are given an initial condition for each equation in (1.1) of the form:

(1.2) \( x_i(0) = x_{i0}, \ i = 1, \ldots, n \)

then the solution to the boundary value problem given by (1.1) and (1.2) (assuming \( a_{ii} \neq 0, i = 1, 2, \ldots, n \)) is:

\[
x_i(t) = \left( x_{i0} + \frac{b_i}{a_{ii}} \right) e^{a_{ii}t} - \frac{b_i}{a_{ii}}, \ i = 1, 2, \ldots, n
\]

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1.2. **Diagonalizable Linear Systems.** If the matrix $A$ is not diagonal, we can use the algebraic technique of diagonalizing a matrix to reduce any diagonalizable linear system to an equivalent diagonal or uncoupled system.

**Theorem 1.** If the matrix $A$ has $n$ linearly independent eigenvectors, $v_1, v_2, \ldots, v_n$, then the matrix $V = (v_1, v_2, \ldots, v_n)$ is invertible and $V^{-1}AV$ is a diagonal matrix with the eigenvalues of $A$ along its main diagonal:

$$V^{-1}AV = \Lambda = \begin{pmatrix} \lambda_{11} & 0 & \ldots & 0 \\ 0 & \lambda_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{nn} \end{pmatrix}$$

Note: If all the eigenvalues of the matrix $A$ are distinct, then the $n$ eigenvectors are linearly independent and therefore $A$ is diagonalizable. If $A$ is diagonalizable, we first derive $x^c(t)$ by solving the homogeneous part of (1.1):

$$x^c(t) = Ax(t) \quad (1.3)$$

$$V^{-1}x^c(t) = V^{-1}Ax(t) = V^{-1}AVV^{-1}x(t) \quad (1.4)$$

where $y(t) = V^{-1}x(t)$. Equation (1.4) is an uncoupled system, which we already know how to solve. Therefore:

$$y_i(t) = c_i e^{\lambda_i t} \quad (1.5)$$

To obtain the solution to the original system we need to revert from $y(t)$ to $x(t)$:

$$x^c(t) = Vy(t) = (v_1, v_2, \ldots, v_n) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t} \quad (1.6)$$

To obtain $x^p(t)$ we use the steady state:

$$x^c(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} \iff Ax(t) + b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

$$x^p(t) = -A^{-1}b, \quad (if \ A \ is \ nonsingular)$$

Therefore:

$$x^p(t) = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t} - A^{-1}b \quad (1.6)$$

The constants $c_1, c_2, \ldots, c_n$ can be found with a set of $n$ boundary conditions.

**Example:** Consider the linear system:

$$x^c(t) = Ax(t) \quad (1.1)$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$
Its eigenvalues and eigenvectors satisfy the equation:
\[(A - \lambda I_2)v = 0\]
(1.7)

The eigenvalues are the roots of the following equation:
\[\det(A - \lambda I_2) = 0\]
\[\begin{vmatrix} 1 - \lambda & 1 \\ 4 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0\]

Thus \(\lambda_1 = 3\) and \(\lambda_2 = -1\). For \(\lambda_1 = 3\), Equation (1.7) becomes:
\[
\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}
\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Then:
\[v_{12} = 2v_{11}\]
and
\[v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]

Similarly, for \(\lambda_2 = -1\) we obtain:
\[v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\]

Therefore:
\[V = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\]

and
\[V^{-1}AV = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\]

Under the transformation:
\[y(t) = V^{-1}x(t)\]
we obtain:
\[y^1(t) = Ay(t) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}
\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}\]

which has the general solution:
\[\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} c_1e^{3t} \\ c_2e^{-t} \end{pmatrix}\]

And the solution to the original system is given by:
\[
\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = Vy(t) =
\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}
\begin{pmatrix} c_1e^{3t} + c_2e^{-t} \\ 2c_1e^{3t} - 2c_2e^{-t} \end{pmatrix}\]
1.3. **Stability.** We can now analyze the stability properties of Equation \((1.6)\).

1.3.1. *All Eigenvalues Real.* We can see from Equation \((1.6)\) that:

- If all eigenvalues are negative, \(e^{\lambda_i t} \to 0\) as \(t \to \infty\), for all \(i = 1, 2, \ldots, n\), and the system is asymptotically stable: it converges to the steady state whatever the initial position.
- If some eigenvalues are positive, the corresponding terms \(e^{\lambda_i t} \to \infty\) as \(t \to \infty\), and \(x_j(t)\) is unstable unless \(c_i v_{ij} = 0\).

1.3.2. *Some Eigenvalues Complex.* If some eigenvalues of \(A\) are complex, then Equation \((1.6)\) is still valid, but it is complex valued. Since we are typically interested only in real valued solutions, we need to transform Equation \((1.6)\) into a real valued expression.

1.4. **Complex Numbers.** A complex number has the form \(a + ib\), where \(a\) and \(b\) are real numbers, and \(i\) is the imaginary unit, \(i = \sqrt{-1}\).

We can represent a complex number as a point with coordinates \((a, b)\) in the *complex plane* in Figure 1.1. The *conjugate* of a complex number \(a + ib\) is the number \(a - ib\), with the sign of the imaginary part reversed. The modulus of a complex number is the norm of the vector that represents it in the complex plane:

\[ r = \sqrt{a^2 + b^2} \]

Let \(\theta\) be the angle formed by the vector representing a complex number and the horizontal axis of the complex plane. Then:

\[ \tan \theta = \frac{b}{a}, \quad \cos \theta = \frac{a}{r}, \quad \sin \theta = \frac{b}{r} \]

and thus

\[ a = r \cos \theta, \quad b = r \sin \theta \]

Therefore:

\[ a + ib = r(\cos \theta + i \sin \theta) \]
By Euler's formula:
\[ e^{i\theta} = \cos \theta + i \sin \theta \]

Then:
\[ a + ib = r(\cos \theta + i \sin \theta) = re^{i\theta} \]

To transform the complex valued solution into a real valued one we will use the fact that if \( A \) is a matrix with real entries, its complex eigenvalues and eigenvectors come in conjugate pairs.

Let us assume that \( \lambda_1 \) and \( \lambda_2 \) are complex and \( \lambda_3, \lambda_4, \ldots, \lambda_n \) are real. Then:
\[ \lambda_1 = \alpha + i\mu, \quad \lambda_2 = \alpha - i\mu \]

and the corresponding eigenvectors:
\[ v_1 = d + if, \quad v_2 = d - if \]

Remember that the general solution was:
\[ x^g(t) = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t} - A^{-1} b \]  

We have:
\[ v_1 e^{\lambda_1 t} = (d + if)e^{(\alpha + i\mu)t} \]
\[ = (d + if)e^{\alpha t}(\cos \mu t + i \sin \mu t) \]
\[ = e^{\alpha t}(d \cos \mu t - f \sin \mu t) + ie^{\alpha t}(f \cos \mu t + d \sin \mu t) \]

And similarly:
\[ v_2 e^{\lambda_2 t} = e^{\alpha t}(d \cos \mu t - f \sin \mu t) - ie^{\alpha t}(f \cos \mu t + d \sin \mu t) \]

Let's define the real-valued terms:
\[ u(t) = e^{\alpha t}(d \cos \mu t - f \sin \mu t) \]
\[ w(t) = e^{\alpha t}(f \cos \mu t + d \sin \mu t) \]

Then, we can rewrite the general solution in Equation (1.8) as:
\[ x^g(t) = c_1 [u(t) + iw(t)] + c_2 [u(t) - iw(t)] + \sum_{i=3}^{n} c_i v_i e^{\lambda_i t} - A^{-1} b \]

Let's choose \( c_1 \) and \( c_2 \) to be complex conjugates of each other:
\[ c_1 = k_1 + ik_2 \quad \text{and} \quad c_2 = k_1 - ik_2 \]

Then, Equation (1.9) becomes:
\[ x^g(t) = \tilde{c}_1 u(t) + \tilde{c}_2 w(t) + \sum_{i=3}^{n} c_i v_i e^{\lambda_i t} - A^{-1} b \]

where \( \tilde{c}_1 = 2k_1 \) and \( \tilde{c}_2 = -2k_2 \). Note that Equation (1.10) is now real valued.

About stability, introduces a cyclical element into the solution through the functions \( \cos \mu t \) and \( \sin \mu t \), but what determines if the system converges or diverges is the term \( e^{\alpha t} \). Therefore, if any eigenvalue is complex, stability will be determined by the real part of the eigenvalue.
If \( n = 2 \), we can use phase diagrams to illustrate the stability properties of the system:

\[
(1.11) \quad x^1(t) = Ax(t) + b
\]

Without loss of generality, let’s assume that \( b = 0 \).

Note: We can always “demean” Equation (1.11) by defining a new variable \( y \) as the deviation of \( x \) from its steady state:

\[
y(t) = x(t) + A^{-1}b
\]

Then:

\[
y^1(t) = x^1(t) = Ax(t) + b = A[y(t) - A^{-1}b] + b = Ay(t)
\]

If \( n = 2 \), all the information about the system’s stability properties is summarized in the trace and the determinant of the coefficient matrix \( A \). The eigenvalues of \( A \) solve:

\[
|A - \lambda I_2| = 0
\]

\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0
\]

\[
\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0
\]

\[
\lambda^2 - tr(A)\lambda + det(A) = 0
\]

Then:

\[
\lambda_{1,2} = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}}{2}
\]

We will also use the following:

\[
tr(A) = \lambda_1 + \lambda_2
\]

\[
det(A) = \lambda_1\lambda_2
\]

The possible cases are:

1.4.1. Case 1: Nodes.

(1) If \( det(A) > 0 \) and \( [tr(A)^2 - 4det(A)] > 0 \): \( \lambda_1 \) and \( \lambda_2 \) are real, distinct, and with the same sign.

- If \( tr(A) < 0 \) \( \Rightarrow \lambda_1, \lambda_2 < 0 \): Stable node (Figure 1.2)
- If \( tr(A) > 0 \) \( \Rightarrow \lambda_1, \lambda_2 > 0 \): Unstable node (Figure 1.3)

(2) If \( [tr(A)^2 - 4det(A)] = 0 \): \( \lambda_1 \) and \( \lambda_2 \) are real and repeated.

- If \( tr(A) < 0 \) \( \Rightarrow \lambda_1 = \lambda_2 < 0 \): Stable node (Figure 1.4)
- If \( tr(A) > 0 \) \( \Rightarrow \lambda_1 = \lambda_2 > 0 \): Unstable node (Figure 1.5)
Figure 1.2. A Stable Node

Figure 1.3. An Unstable Node

Figure 1.4. A Stable Node with a single eigenvector
1.4.2. Case 2: Saddle points. If \( \det(A) < 0 \): \( \lambda_1 \) and \( \lambda_2 \) are real, distinct, and with opposite sign.

For concreteness, assume that \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \). Then, if \( c_1 v_{1j} \neq 0 \), \( x_j(t) \) is unstable. (Note that \( v_1 \) can not be the zero vector for \( V \) to be nonsingular). But if \( c_1 = 0 \), the system’s behavior is determined only by \( \lambda_2 \), and it converges to the steady state as \( t \to \infty \). Setting \( c_1 = 0 \) we obtain:

\[
x_1(t) = c_2 v_{21} e^{\lambda_2 t}
\]

and

\[
x_2(t) = c_2 v_{22} e^{\lambda_2 t}
\]

Then:

\[
x_1(t) = \frac{v_{12}}{v_{22}} x_2(t)
\]

which is the equation of the saddle path.

For the following graph in Figure 1.6, assume that

\[
v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

Then:

\[
A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

Hence, the system will converge to the steady state if and only if it starts out from one of the points where the constant associated with the explosive root is equal to zero.

1.4.3. Case 3: Spiral Points. If \( [tr(A)^2 - 4\det(A)] < 0 \) and \( tr(A) \neq 0 \): \( \lambda_1 \) and \( \lambda_2 \) are complex conjugates.

The circular functions \( \cos \mu t \) and \( \sin \mu t \) in the solution introduce a spiral-like pattern. It can be shown that the spirals are clockwise if \( a_{21} < 0 \) and counterclockwise if \( a_{21} > 0 \). Spirals can be either stable or unstable. That can be determined by the sign on the trace:

\[
tr(A) = \lambda_1 + \lambda_2 = \alpha + i\mu + \alpha - i\mu = 2\alpha
\]

Then:
Figure 1.6. A Saddle Point

Figure 1.7. A Stable Spiral

- If $tr(A) < 0$: The eigenvalues have a negative real part and the spirals converge to the steady state. Suppose $a_{21} > 0$, then a potential phase diagram could be given by Figure 1.7.

- If $tr(A) > 0$: The eigenvalues have a positive real part and the spirals diverge from the steady state. Suppose $a_{21} > 0$ then a potential phase diagram could be given by Figure 1.8.
1.4.4. Case 4: Centers. If $tr(A) = 0$ and $\det(A) > 0$: $\lambda_1$ and $\lambda_2$ are pure imaginary numbers. The steady state is stable but not asymptotically. Suppose $a_{21} > 0$, then a trajectories could be given by Figure 1.9.

1.4.5. Non-Diagonalizable Linear Systems. If the matrix $A$ is not diagonalizable, we need to use other method to find the solution. See, for example, Perko (1991) for a treatment of these cases.

1.5. Elements of Nonlinear Systems. In general, it is not possible to find closed-form solutions for nonlinear differential equations. However, we can obtain qualitative information about the local behavior of the solution using graphical analysis if $n = 2$, and study nonlinear systems analytically by looking at their linearizations.
Consider the nonlinear differential equation system:

\[
\begin{align*}
\dot{x}_1(t) &= F_1[x_1(t), x_2(t)] \\
\dot{x}_2(t) &= F_2[x_1(t), x_2(t)]
\end{align*}
\]  

Phase Diagrams: Assume that \( n = 2 \):

1. Step 1: Set \( x_1(t) = 0 \) and \( x_2(t) = 0 \) to obtain the phase lines.

\[
\begin{align*}
\dot{x}_1(t) &= f_1[x_1(t), x_2(t)] \\
\dot{x}_2(t) &= f_2[x_1(t), x_2(t)]
\end{align*}
\]

Each of these equations describes a curve in the \((x_1(t), x_2(t))\) plane.

- Phase line \( x_1(t) = 0 \):
- Phase line \( x_2(t) = 0 \):
  
  These isoclines are presented in Figures 1.10 and 1.11.

2. Step 2: Use equations (1.13) and (1.14) to obtain the arrows of motion.

To determine whether \( x_1(t) \) increases or decreases to the left (right) of the
Figure 1.12. Phase Diagram for a non-linear differential equation.

Phase line, we will evaluate either of the derivatives:

\[
\frac{\partial x_1(t)}{\partial x_1(t)} = \frac{\partial f_1[x_1(t); x_2(t)]}{\partial x_2(t)}
\]

\[
\frac{\partial x_2(t)}{\partial x_2(t)} = \frac{\partial f_1[x_1(t), x_2(t)]}{\partial x_2(t)}
\]

at a point in the line \( x_1(t) = 0 \) (typically at the steady state). For example, suppose that:

\[
\left. \frac{\partial x_1(t)}{\partial x_1(t)} \right|_{x_1(t)=0} = \left. \frac{\partial f_1[x_1(t), x_2(t)]}{\partial x_1(t)} \right|_{x_1(t)=0} < 0
\]

That means that, starting from the \( x_1(t) = 0 \) line, a small movement to the right will decrease \( x_1(t) \), making it strictly negative. Thus, to the right of \( x_1(t) = 0 \), \( x_1(t) < 0 \) and \( x_1(t) \) is decreasing. The opposite happens to the left of \( x_1(t) = 0 \). Now, do the same for \( x_2(t) \). Suppose that:

\[
\left. \frac{\partial x_2(t)}{\partial x_2(t)} \right|_{x_2(t)=0} = \left. \frac{\partial f_2[x_1(t), x_2(t)]}{\partial x_2(t)} \right|_{x_2(t)=0} < 0
\]

That means that, starting from the \( x_2(t) = 0 \) line, a small movement to upward will decrease \( x_2(t) \), making it strictly negative. Thus, up from \( x_2(t) = 0 \), \( x_2(t) < 0 \) and \( x_2(t) \) is decreasing. The opposite happens down from \( x_2(t) = 0 \).

(3) Step 3: Combine the two previous graphs, yielding something like Figure 1.12.

Although we can obtain valuable information about the behavior of the system, in this case, for example, the phase diagram by itself does not give us enough information about the stability properties of the steady state.

To obtain more precise (but only local) information about the behavior of nonlinear systems, under certain conditions we can approximate the systems by their linear counterparts.

Take a multivariate first-order Taylor series approximation of \( F \) around \( \bar{x} \):

\[
F[x(t)] = F(\bar{x}) + DF[x(t)][x(t) = \bar{x}] \cdot |x(t) - \bar{x}| + R|x(t) - \bar{x}|
\]
where

$$DF[x(t)]|_{x(t)=\bar{x}} = \left( \begin{array}{cccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{array} \right)_{x(t)=\bar{x}}$$

and

$$\lim_{x_i(t)\to \bar{x}_i} \frac{\|R[x_i(t) - \bar{x}_i]\|}{\|x_i(t) - \bar{x}_i\|} = 0, \ i = 1, 2, \ldots, n$$

That is, in some neighborhood of the steady state, the linear system:

$$x^1(t) = DF[x(t)]|_{x(t)=\bar{x}} \cdot [x(t) - \bar{x}]$$

can be expected to be a reasonable approximation of Equation (1.12).

The condition under which this is true is again that the steady state must be a hyperbolic equilibrium.

**Definition 2.** (Hyperbolic Equilibrium). Let $\bar{x}$ be a steady state of the nonlinear system Equation (1.12). We say that $\bar{x}$ is a hyperbolic equilibrium if the derivative of $F$ evaluated at $\bar{x}$, $DF[x(t)]|_{x(t)=\bar{x}}$ has no eigenvalues with a zero real part.

**Theorem 3.** (Grobman-Hartman Theorem). If $\bar{x}$ is a hyperbolic equilibrium of the autonomous differential equation system

$$x^1(t) = F[x(t)], \ F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n, \ F \in C^1$$

then there exists a neighborhood $U$ of $\bar{x}$ such that Equation (1.15) is topologically equivalent to the linear system

$$x^1(t) = DF[x(t)]|_{x(t)=\bar{x}} \cdot [x(t) - \bar{x}]$$

in $U$. 